Lower Bounds for the Classical Scheduling Problem

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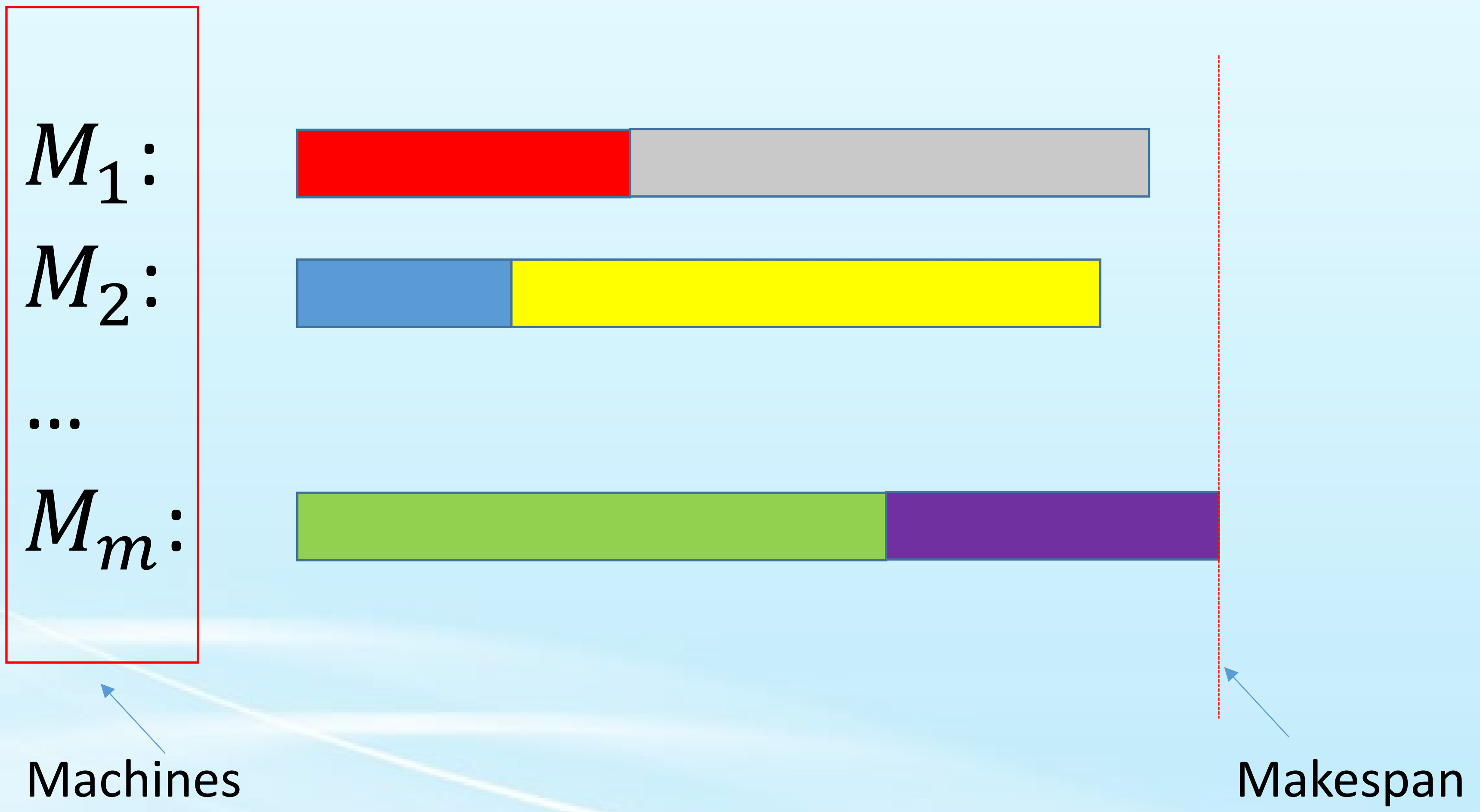
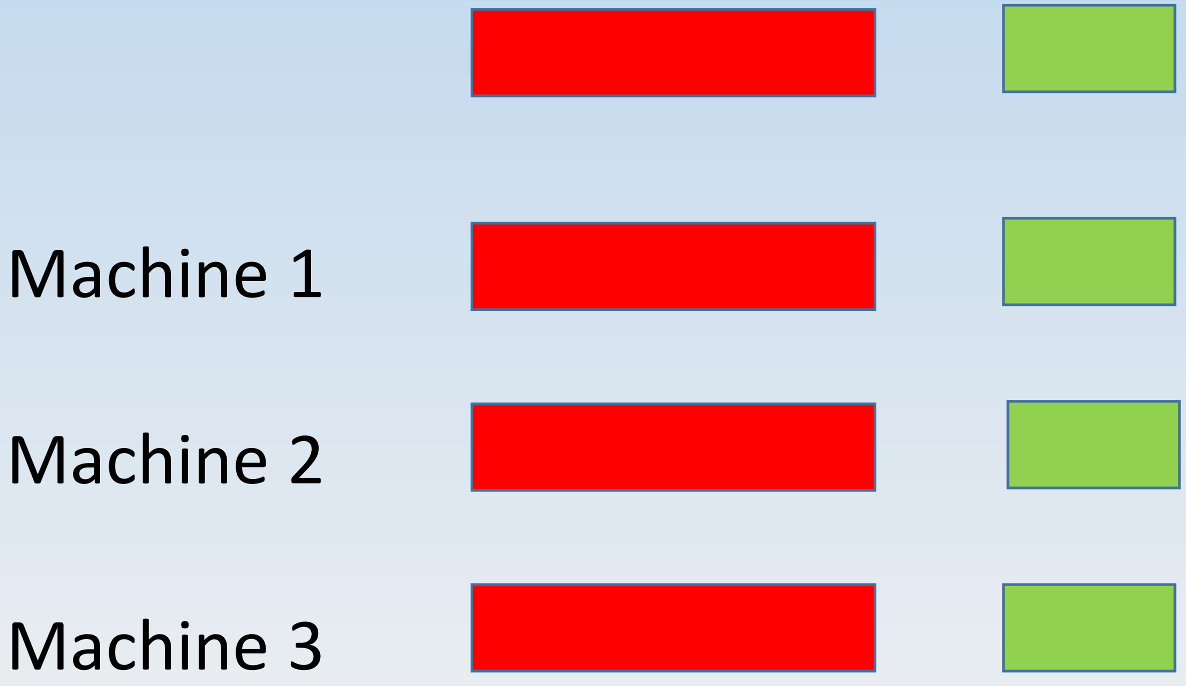
The classical scheduling problem



Identical machines P//Cmax



Related machines Q//Cmax

Machine 1 (speed=1/2)

Machine 2 (speed=1)

Machine 3 (speed=2)

Unrelated machines R//Cmax

Known results

Approximation results

- $LPT\ (Longest\ Proocessing\ Time)$: $\frac{4}{3} - \frac{1}{3m}$ for P//Cmax
- $PTAS$ for Q//Cmax
- $FPTAS$ for Rm//Cmax
- $\left(2 - \frac{1}{m}\right)$ -approximation for R//Cmax
- $(\frac{3}{2} - \epsilon)$- approximation for R//Cmax implies P=NP

Two topics

- Improving running time for approximation schemes

- Improving performance rations for R//Cmax

Lower bounds on running time

Approximation schemes for $P||C_{max}$

- $(n/\epsilon)^{O(1/\epsilon^2)}$ (Hochbaum and Shmoys, JACM 1987)
- $f(1/\epsilon) + O(n)$ doubly exponential in $1/\epsilon$ (Alon et al., SODA 1998)
- $2^{O(1/\epsilon^2 \log^3 1/\epsilon)} + n^{O(1)}$ (Jansen, SIAM J. Discrete Mathematics 2010)

Approximation schemes for $Pm||C_{max}$

- $O(nm(nm/\epsilon)^{m-1})$ (Horowitz and Sahni, JACM 1976)
- $(n\ +\ 1)^{m/\epsilon}\,poly(|I|)$ (Lenstra, Shmoys, Tardos, Math. Prog 1990)
- $n(m/\epsilon)^{O(m)}$ (Jansen and Porkolab, MOR 2001)
- $O(n)\ +\ (1/\epsilon)^{O(m)}$ when $1/\epsilon > m$ (Jansen and Mastrolilli, 2010)

Approximation schemes for $Rm||C_{max}$

- $O(nm(nm/\epsilon)^{m-1})$ (Horowitz and Sahni, JACM 1976)
- $(n\ +\ 1)^{m/\epsilon}\ poly(|I|)$ (Lenstra, Shmoys, Tardos, Math. Prog 1990)
- $n(m/\epsilon)^{O(m)}$ (Jansen and Porkolab, MOR 2001)
- $O(n)\ +\ (1/\epsilon)^{O(m)}$ when $1/\epsilon > m$ (Jansen and Mastrolilli, 2010)

Our contribution

Algorithms	Upper bounds	Lower bounds
Approximation schemes	$2^{O(1/\epsilon^2 \log^3 1/\epsilon)} + \mathrm{n}^{O(1)}$	$2^{O((1/\epsilon)^{1-\delta})} + \mathrm{n}^{O(1)}$
Approximation schemes	$(1/\epsilon)^{O(m)} + O(n)$	$(1/\epsilon)^{O(m^{1-\delta})} + n^{O(1)}$
Exact algorithm	$2^{O(\sqrt{m\|I\| \log \quad m})}$	$2^{O(m^{1/2-\delta}\sqrt{\|I\|})}$
Exact algorithm	$2^{O(n)}$	$2^{O(n^{1-\delta})}$

Based on Exponential Time Hypothesis (ETH), assuming that 3SAT could not be solved in 2^{sn} time for some s>0 (Impagliazzo, Paturi, Zane, 2001)

A small gap for $P||C_{max}$

- $2^{O(1/\epsilon^2 \log^3 1/\epsilon)} + \mathrm{n}^{O(1)}$ $vs.$ $2^{O((1/\epsilon)^{1-\delta})} + \mathrm{n}^{O(1)}$

There exists a $2^{O(1/\epsilon \log^2 1/\epsilon)} + \mathrm{n}^{O(1)}$ - time algorithm,

(1) if every machine can accommodate at most a constant number of jobs;

(2) if a certain conjecture holds (Jansen and Robenek, WAOA 2011)

Reduction for $P||C_{max}$

- 2^{sn} for 3SAT → $2^{O((1/\epsilon)^{1-\delta})}$ for Scheduling

Goal

- 3SAT instance of $O(n)$ variables and clauses

→ A scheduling instance of makespan $L = O\left(n^k\right)$ ⇒ $\boldsymbol{L = O(n^{1+\delta})}$

A lower bound of $2^{O((1/\epsilon)^{1/k-\delta})}$ by taking $1/\epsilon = L + 1 = O(n^k)$
(A $(1+\epsilon)$-alg. is able to distinguish the makespan of L and L+1)

$k = 16$ in the traditional reduction

Reduction for $Pm||C_{max}$

- 2^{sn} for 3SAT → $(1/\epsilon)^{O(m^{1-\delta})}$ for Scheduling

Goal

- 3SAT instance of $O(n)$ variables and clauses

→ A scheduling instance of makespan $\mathrm{L} = 2^{O(\frac{n}{m}\log^2 m)}$

Taking $1/\epsilon = L + 1$, A $(1+\epsilon)$-algorithm is able to distinguish the makespan of L and L+1, running in $2^{\delta_m n}$ time with $\delta_m \to 0$ when $m \to \infty$

Open problems

- Close the gap between $2^{O(1/\epsilon^2 \log^3 1/\epsilon)} + \mathrm{n}^{O(1)}$ *and* $2^{O((1/\epsilon)^{1-\delta})} + \mathrm{n}^{O(1)}$ for $P||C_{max}$

- Influence of additional constraints: $P|r_j|C_{max}$

- Lower bounds on running times of approximation schemes for other classical objectives: $P||\sum w_j C_j$, $Q||\sum w_j C_j$, $P||\sum C_i^p$

Lower bound
for lower rank scheduling

Matrix of job processing times

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1,m} \\ p_{21} & p_{22} & \cdots & p_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n,1} & p_{n,2} & \cdots & p_{n,m} \end{bmatrix}$$

More precisely, the rank d comes from:

$$\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1,m} \\ p_{21} & p_{22} & \cdots & p_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ p_{n,1} & p_{n,2} & \cdots & p_{n,m} \end{bmatrix} = \underbrace{\begin{bmatrix} * & * \\ * & * \\ \vdots & \vdots \\ * & * \end{bmatrix}}_{n * d} \underbrace{\begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \end{bmatrix}}_{d * m}$$

Rank of scheduling

- Identical machines: Rank=1
- Related machines: Rank=1
- Unrelated machines: Rank could be arbitrary

The hardness of the problem and the rank of matrix?

Bhaskara, Krishnaswamy, Talwar, Wieder, SODA 2013

Rank of Matrix	Hardness
1	PTAS (Hauchbaum and Shmoys)
2	QPTAS
3	open
4	APX-hard
≥7	(3/2-ϵ)-inapproximate

New lower bounds

Rank of Matrix	Hardness
3	APX-hard
≥4	(3/2-∊)-inapproximate

Rank 4 scheduling

Reduction from 3DM

- ***Element sets***: $W \cup X \cup Y, |\mathrm{W}| = |\mathrm{X}| = |\mathrm{Y}|$
- ***Match set***: $T \subset W \times X \times Y$
- ***Perfect matching***: $T' \subset T, every\ element\ appears\ once$

NP-hard to decide whether there exists a perfect matching

Rank 7 scheduling

Bhaskara, et al. 2013

- One machine for every match:
  $(w_i, x_j, y_k) \to (N^i, N^{-i}, N^j, N^{-j}, N^k, N^{-k}, 1)$

- Real jobs:
  $x_j \to (\ 0,\quad 0,\quad \epsilon N^{-j},\quad \epsilon N^j,\quad 0,\quad 0,\quad 1)$
  $y_k \to (\ 0,\quad 0,\quad 0,\quad 0,\quad \epsilon N^{-k}\quad \epsilon N^k,\ 1)$

- Dummy jobs: $d(w_i) - 1$ copies
  $w_i \to (\epsilon N^{-i}, \epsilon N^i,\quad 0,\quad 0,\quad 0,\quad 0,\quad 2)$

Rank 7 scheduling

Bhaskara, et al. 2013

- The processing time of each real job is at least 1
- The processing time of a dummy job is at least 2

  -- Ensured by the 7-th coordinate

- A w-job has to be scheduled on a machine (match) containing element w

  -- Ensured by the first 2 coordinates

Rank 7 scheduling

Bhaskara, et al. 2013

A schedule of length $2 + O(\epsilon)$

$(w_1,*,*)$ $w_1 - dummy$

$(w_1,*,*)$ $w_1 - dummy$

$\dots$

(w_1, x_j, y_k) $x_j - job, y_k - job$

$(w_2,*,*)$ $w_2 - dummy$

$\dots$

Rank 4 scheduling

$$LRS(7) \leftarrow (*, \quad *, \quad *, \quad *, \quad *, \quad *, \quad *)$$

$w - coordinates$ $\quad$ $x - coordinates$ $\quad$ $y - coordinates$

Let w, x, y share coordinates

However, in the 3DM problem, for a match (w_i, x_j, y_k), the indices (i,j,k) could be arbitrary

A variant of 3DM

Reduction from a special 3DM (still NP-complete)

- Element sets: $W = \{w_i, \bar{w}_i | i \in [3n]\}, X = \{s_i, a_i | \ i \in [3n]\}, Y = \{s'_i, b_i | \ i \in [3n]\}$
- Match set: $T_1 \subset \left\{ \left(w_i, s_j, s'_j\right), \left(\bar{w}_i, s_j, s'_j\right) \middle| w_i \in W, s_j \in X, s'_j \in Y \right\}$
  $T_2 = \{(w_i, a_i, b_i), (\bar{w}_i, a_i, b_{\zeta(i)}) | \ i \in [3n]\}$

$\zeta: \zeta(3k+1) = 3k+2, \zeta(3k+2) = 3k+3, \zeta(3k+3) = 3k+1$

Tied indices

(w_i, s_j, s'_j), $(\bar{w}_i, s_j, s'_j)$, (w_i, a_i, b_i), $(\bar{w}_i, a_i, b_{\zeta(i)})$

Indices are tied

Overall structure

Machines

- (w_i, s_j, s_j'): $(N^i, N^{-i}, N^{j+N}, N^{-j-N})$
- $(\bar{w}_i, s_j, s_j')$: $(N^{-i}, N^i, N^{j+N}, N^{-j-N})$
- $(w_i, a_i, *)$: $(N^i, N^{-i}, N^{-i}, *)$
- $(\bar{w}_i, a_i, *)$: $(N^{-i}, N^i, N^{-i}, *)$

Jobs

- w_i: $(N^{-i}, N^i, 0, 0)$
- $\bar{w}_i$: $(N^i, N^{-i}, 0, 0)$
- a_i: $(N^{-i}, N^{-i}, N^i, 0)/2$
- $s_j(s_j')$: $(0, 0, N^{-j-N}, N^{j+N})/2$

Overall structure

Machines

- (w_{3i}, a_{3i}, b_{3i}): $\left(N^{3i}, \quad N^{-3i}, \quad N^{-3i}, \quad N^{-3i-1}\right)$
- $(w_{3i+1}, a_{3i+1}, b_{3i+1})$: $\left(N^{3i+1}, \quad N^{-3i-1}, \quad N^{-3i-1}, \quad N^{-3i}\right)$
- $(w_{3i+2}, a_{3i+2}, b_{3i+2})$: $\left(N^{3i+2}, \quad N^{-3i-2}, \quad N^{-3i-2}, \quad \frac{1}{\epsilon}N^{-3i-1}\right)$
- $(\overline{w}_{3i}, a_{3i}, b_{3i+1})$: $\left(N^{-3i}, \quad N^{3i}, \quad \epsilon N^{-3i}, \quad N^{-3i}\right)$
- $(\overline{w}_{3i+1}, a_{3i+1}, b_{3i+2})$: $\left(N^{-3i-1}, \quad N^{3i+1}, \quad N^{-3i-1}, \quad \frac{1}{\epsilon}N^{-3i-1}\right)$
- $(\overline{w}_{3i+2}, a_{3i+2}, b_{3i})$: $\left(N^{-3i-2}, \quad N^{3i+2}, \quad \epsilon N^{-3i-2}, \quad N^{-3i-1}\right)$

Jobs

- b_{3i}: $\left(\epsilon N^{-3i}, \quad \epsilon N^{-3i-2}, \quad N^{3i}, \quad N^{3i+1}\right)/2$
- b_{3i+1}: $\left(\epsilon N^{-3i-1}, \quad \frac{1}{\epsilon}N^{-3i-1}, \quad N^{3i}, \quad N^{3i}\right)/2$
- b_{3i+2}: $\left(\epsilon N^{-3i-2}, \quad \epsilon N^{-3i-1}, \quad N^{3i+1}, \quad \epsilon N^{3i+1}\right)/2$

Gap

- If the instance of the special 3DM has a perfect matching, then there exists a schedule with a makespan of $2+O(\epsilon)$

- If the scheduling instance has a feasible schedule of makespan smaller than 3, then the 3DM has a perfect matching

- This gives a lower bound of $3/2-\epsilon$

Open problems

. Approximation algorithms for scheduling problem of a constant rank.
. Is rank 2 scheduling harder than rank 1 (in terms of schemes)?
. Is rank 4 scheduling harder than rank 3 (in terms of approx ratios)?

Thank You!